An Analysis of Two Cascaded Fourier Transforms and Application to the Coarse and Fine PFBs for the MWA

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Herein I discuss certain aspects of cascaded Fourier transforms such as those computed in the coarse and fine polyphase filter boards (PFBs) in the MWA-LFD instrument. Before the analysis of the PFBs is presented, I first go through the mathematics of cascaded FFTs without the application of window functions in either stage. Even though this does not model the MWA data processing fully, it retains the essential points.

The Cascaded Fourier Transform

Data, which may be taken as a stream of real or complex equally-spaced measurements, are divided into equal-size sets and transformed. This first-stage or coarse transform produces complex Fourier coefficients for each of a set of equally spaced frequencies. For each of a set of selected coarse channels the coefficients from the succession of coarse data sets are formed into a time series, and when the series is complete, a transform is done on them. This is the second stage or fine transform. In essence, the first stage analyzes the input data stream into coarse frequency channels and then the second stage transform analyzes each selected coarse channel into fine frequency channels.

We use index i to sequentially number the n measurements within each time interval over which the first-stage (coarse) transform operates and index s to number the M first-stage intervals that go into the second stage (fine) transform. The coarse transform for interval s results in a complex Fourier coefficient for frequency bin k that may be expressed as

$$\tilde{d}_{k,s} = \sum_{i=0}^{n-1} d_{sn+i} e^{j\omega_k t_i}. \tag{1}$$

Here, $t_i = i\Delta t$ is the elapsed time from the beginning of this coarse interval (interval s) to measurement i and the angular frequency is $\omega_k = 2\pi k/(n\Delta t)$. The data points are numbered consecutively starting at zero from the beginning of the first coarse interval. A total of M complex Fourier coefficients for coarse channel k are collected and transformed in the fine transform to give the Fourier coefficient for fine channel l according to

$$\begin{aligned}
\tilde{D}_{kl} &= \sum_{s=0}^{M-1} \tilde{d}_{k,s} e^{j\omega_l t_{sn}} && (2)\\
&= \sum_{s=0}^{M-1} \left(\sum_{i=0}^{n-1} d_{sn+i} e^{j\omega_k t_i} \right) e^{j\omega_l t_{sn}} && (3)\\
&= \sum_{s=0}^{M-1} \sum_{i=0}^{n-1} d_{sn+i} e^{j(\omega_k t_i + \omega_l t_{sn})}. && (4)
\end{aligned}$$

In this expression, $t_{sn} = sn\Delta t$ is the elapsed time from the beginning of the first coarse interval to the beginning of coarse interval s and the frequency is $\omega_l = 2\pi l/(Mn\Delta t)$.

It is now useful to compare the above expression for $\tilde{D}_{kl}$ with the result that would be obtained from a Fourier transform of the whole set of $N = nM$ measurements. In the latter case one obtains

$$\tilde{F}_p = \sum_{s=0}^{M-1} \sum_{i=0}^{n-1} d_{sn+i} e^{j\omega_p (t_{sn}+t_i)} \tag{5}$$

where $\omega_p = 2\pi p/(Mn\Delta t)$. One may set $p = kM + l$, so that $\omega_p = \omega_k + \omega_l$. Then one finds:

$$\tilde{F}_p = \tilde{F}_{kl} = \sum_{s=0}^{M-1} \sum_{i=0}^{n-1} d_{sn+i} e^{j(\omega_k+\omega_l)(t_{sn}+t_i)}. \tag{6}$$

One is free to insert the factor $e^{j\omega_k t_{sn}}$ into the expression for $\tilde{D}_{kl}$ because this factor equals unity for any value of k and any value of s since the exponent is an integral multiple of $2\pi j$. We can therefore express $\tilde{D}_{kl}$ as

$$\tilde{D}_{kl} = \sum_{s=0}^{M-1} \sum_{i=0}^{n-1} d_{sn+i} e^{-j\omega_l t_i} e^{j(\omega_k+\omega_l)(t_{sn}+t_i)}. \tag{7}$$

This differs from the expression for $\tilde{F}_{kl}$ by the factor $e^{-j\omega_l t_i}$ which is a function of time that depends on l but not on k and is periodic with the period $n\Delta t$.

Action on an Exponential Function

One may expand any set of real- or complex-valued measurements obtained from $N = nM$ equally-spaced times in terms of complex exponentials with the angular frequencies $\omega = 2\pi(kM + l)/(N\Delta t)$ $(0 < k \leq n-1,\ 0 < l \leq M-1)$. (If the measurements are real-valued, then one may dispense with the frequencies having $n/2 \leq k \leq n-1$.) We may therefore assume that the measured data points can be written as

$$d_{sn+i} = a e^{-j(\omega_{k'}+\omega_{l'})(t_{sn}+t_i)+\phi)} \tag{8}$$

The primes on k' and l' distinguish those symbols from the indices of the frequency channels of the data coming out from the transforms. I now take this expression, omit the constant factors $(a/2)$ and $e^{-j\phi}$, and insert it into the expression for $\tilde{D}_{kl}$. The result is denoted by a new symbol:

$$\begin{aligned}
\tilde{T}_{kl} &= \sum_{s=0}^{M-1} \sum_{i=0}^{n-1} e^{-j(\omega_{k'}+\omega_{l'})(t_{sn}+t_i)} e^{-j\omega_l t_i} e^{j(\omega_k+\omega_l)(t_{sn}+t_i)} && (9)\\
&= \sum_{s=0}^{M-1} \sum_{i=0}^{n-1} e^{j[(\omega_k-\omega_{k'}+\omega_l-\omega_{l'})t_{sn}+(\omega_k-\omega_{k'}-\omega_{l'})t_i]} && (10)\\
&= \left(\sum_{s=0}^{M-1} e^{j(\omega_k-\omega_{k'}+\omega_l-\omega_{l'})t_{sn}}\right)\left(\sum_{i=0}^{n-1} e^{j(\omega_k-\omega_{k'}-\omega_{l'})t_i}\right) && (11)
\end{aligned}$$

Consider the first sum on the right side. It is

$$\sum_{s=0}^{M-1} e^{j(\omega_k - \omega_{k'} + \omega_l - \omega_{l'})t_{sn}} = \sum_{s=0}^{M-1} e^{2\pi j[(k-k')sn/n + (l-l')s/M]} \tag{12}$$

$$= \sum_{s=0}^{M-1} e^{2\pi j(l-l')s/M} \tag{13}$$

where the last equality follows because $(k-k')s$ is an integer. If $l = l'$, we then have

$$S_1 = \sum_{s=0}^{M-1} e^{2\pi j(l-l')s/M} = M \tag{14}$$

whereas if $l \neq l'$, we have

$$S_1 = \sum_{s=0}^{M-1} e^{2\pi j(l-l')s/M} = 0. \tag{15}$$

The second sum on the right side is

$$\sum_{i=0}^{n-1} e^{j(\omega_k - \omega_{k'} - \omega_{l'})t_i} = \sum_{i=0}^{n-1} e^{2\pi j[(k-k')/n - l'/(nM)]i} \tag{16}$$

If $k = k'$ and $l' = 0$ then

$$\sum_{i=0}^{n-1} e^{2\pi j[(k-k')/n - l'/(nM)]i} = n. \tag{17}$$

Otherwise, i.e., if $l' \neq 0$ or $k \neq k'$, then

$$S_2 = \sum_{i=0}^{n-1} e^{2\pi j[(k-k')/n - l'/(nM)]i} \tag{18}$$

$$= \frac{1 - e^{2\pi j[(k-k')/n - l'/(nM)]n}}{1 - e^{2\pi j[(k-k')/n - l'/(nM)]}} \tag{19}$$

$$= \frac{1 - e^{2\pi j[(k-k') - l'/M]}}{1 - e^{2\pi j[(k-k')/n - l'/(nM)]}} \tag{20}$$

$$= \frac{1 - e^{-2\pi j l'/M}}{1 - e^{2\pi j[(k-k')/n - l'/(nM)]}} \tag{21}$$

$$= -e^{\pi j(n-1)[(k-k')/n - l'/(nM)]} \frac{\sin \pi l'/M}{\sin \pi[(k-k')/n - l'/(nM)]}. \tag{22}$$

In the case that $l' \neq 0$ but $k = k'$, then

$$S_2 = e^{-\pi j(n-1)l'/(nM)]} \frac{\sin \pi l'/M}{\sin \pi l'/(nM)}. \tag{23}$$

Aliases

The above analysis shows that the only signatures of a sinusoidal signal with a frequency $\omega = 2\pi(k'M + l')/(N\Delta t)$ appear in the fine frequency channels of index $l = l'$ of one or more coarse frequency channels. If $l' = 0$, the signature of the signal appears only in coarse channel $k = k'$ and fine channel $l = l'$. If $l' \neq 0$ the strongest signature appears in coarse channel $k = k'$ while weaker signatures appear in other coarse channels with $k \neq k'$. The ratio of the absolute values of the Fourier amplitudes is

$$\frac{\mathrm{Ampl(k \neq k')}}{\mathrm{Ampl(k = k')}} = \frac{\sin \pi l'/(nM)}{\sin \pi((k-k')/n - l'/(nM))} \tag{24}$$

The strength of the aliases depends on the value of l' and is maximum for the highest absolute values of l', i.e., $l' = \pm M/2$ (to be exact, the fine transform results will be interpreted in terms of $-M/2 + 1 \leq l' \leq M/2$).

Analysis of Cascaded Digital Polyphase Filter Banks

The case of cascaded polyphase filter banks (PFB's) is analyzed below in a manner similar to that of cascaded straight FFT's presented above. The essential difference from the straight FFT case is the use of window functions. The first stage PFB produces Fourier amplitudes which may be expressed as

$$\tilde{d}_{k,s} = \sum_{r=0}^{N_{T1}-1} \sum_{i=0}^{n-1} W_{1,rn+i} d_{sn+rn+i} e^{j\omega_k t_i} \tag{25}$$

where $W_{1,rn+i}$ is the value of the window function in time bin $rn + i$ and N_{T1} is the number of taps in this PFB. The double summation can be expressed as a single sum over the index $u = rn + i$:

$$\tilde{d}_{k,s} = \sum_{u=0}^{nN_{T1}-1} W_{1,u} d_{sn+u} e^{j\omega_k t_u}. \tag{26}$$

The substitution of $t_u = u\Delta t$ for t_i (or vice versa) is valid here because $\omega_k(t_u - t_i)$ is an integral multiple of 2π. The second stage PFB takes series of these complex amplitudes for each value of k and computes the transform according to:

$$\tilde{D}_{k,l} = \sum_{b=0}^{N_{T2}-1} \sum_{s=0}^{M-1} W_{2,bM+s} \tilde{d}_{k,bM+s} e^{j\omega_l t_{sn}}. \tag{27}$$

Again, the double summation can be expressed as a single sum, in this instance over the index $v = bM + s$:

$$\tilde{D}_{k,l} = \sum_{v=0}^{MN_{T2}-1} W_{2,v}\tilde{d}_{k,v}e^{j\omega_l t_{vn}} \tag{28}$$

where, as above, the substitution of t_{vn} in place of t_{sn} (or vice versa) is completely valid. This may be expanded to obtain

$$\tilde{D}_{k,l} = \sum_v \sum_u W_{2,v}W_{1,u}d_{vn+u}e^{j(\omega_k t_u + \omega_l t_{vn})}. \tag{29}$$

Now, we take the input time series to be a complex exponential, i.e.,

$$d_{vn+u} \quad = \quad e^{-j(\omega_{k'}+\omega_{l'})t_{vn+u}} \tag{30}$$

where, as above, $\omega_{k'} = 2\pi k'/(n\Delta t)$ and $\omega_{l'} = 2\pi l'/(Mn\Delta t)$. After this is substituted into the expression for $\tilde{D}_{k,l}$ and the result is simplified and factored, one obtains

$$\tilde{D}_{k,l} \quad = \quad \left[\sum_u W_{1,u}e^{2\pi j[(k-k')/n - l'/(Mn)]u}\right]\left[\sum_v W_{2,v}e^{2\pi j[(l-l')/M-k']v}\right] \tag{31}$$

The first factor on the right hand side is complicated to analyze while the analysis of the second factor is straightforward. The window functions that, to my present knowledge, are used in the 32T MWA-LFD system are shown in Figures 1 and 2. To proceed with the analysis of the second factor, the single sum over v is converted back to a double sum per the original expressions for $\tilde{D}_{k,l}$. Then we have

$$S_v = \quad = \quad \sum_v W_{2,v}e^{2\pi j[(l-l')/M-k']v} \tag{32}$$

$$= \quad \sum_{b=0}^{N_{T2}-1}\sum_{s=0}^{M-1} W_{2,bM+s}e^{2\pi j[(l-l')/M-k'](bM+s)} \tag{33}$$

$$= \quad \sum_s \left(\sum_b W_{2,bM+s}\right) e^{2\pi j[(l-l')/M-k']s} \tag{34}$$

The sum in parentheses in the last expression is, for the MWA fine PFB window function, approximately a constant function of s (see Figure 3). Then the sum is very small except when $l = l'$, i.e., when the fine channel numbers are identical. This applies even when the signal is centered in a different coarse channel. This is illustrated in Figure 4 by the results of a simulation of the action of the MWA-LFD 32T PFBs. The simulations are computed in double precision and do not yet comprehensively model the limited numbers of bits used for the representations of various quantities in the MWA firmware. However, the window

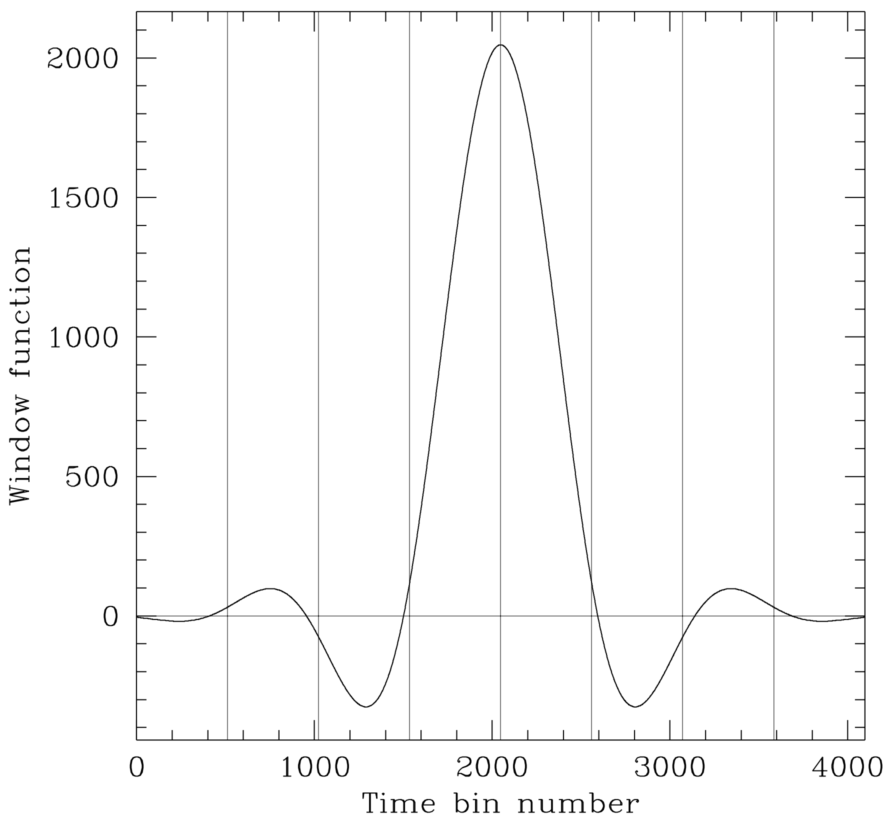

Figure 1: Window function intended for use in the MWA coarse PFB as defined in the file "MwaPfbProtoFilterCoeff2009_512x8.dat". The vertical lines distinguish the intervals corresponding to the 8 taps that are stacked before the FFT is computed.

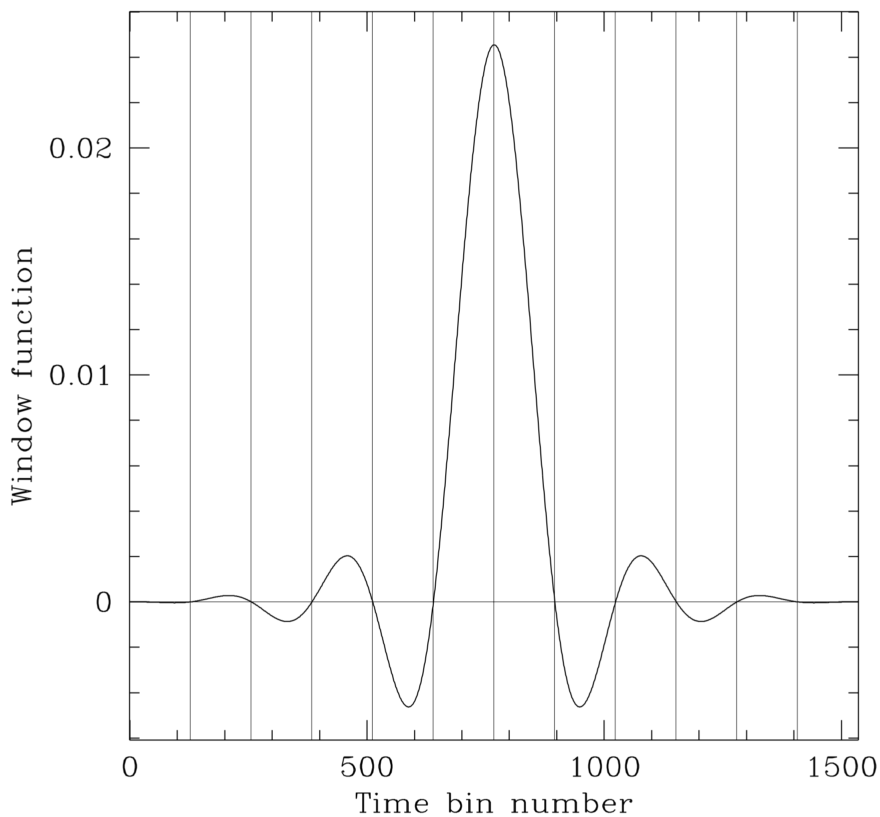

Figure 2: Window function used in the MWA fine PFB. The vertical lines show the intervals corresponding to the 12 taps that are stacked before the FFT is computed.

function shown in Fig. 1 consists of integer values in the range -2047 to 2047. I have found that this limited precision results in additional small alias responses, such as the “secondary” peaks apparent in the two panels on the right in Fig. 4.

The overall power response of the fine channels within a coarse channel is shown in Figure 5 for two possible coarse PFB window functions. The solid curve shows the response computed using the window function shown in Fig. 1. The dashed curve shows the response computed using a slightly narrower window function, i.e., a window function that gives a half-power width closer to 1.28 MHz. Additional discussion of these simulations will be presented in a separate document.

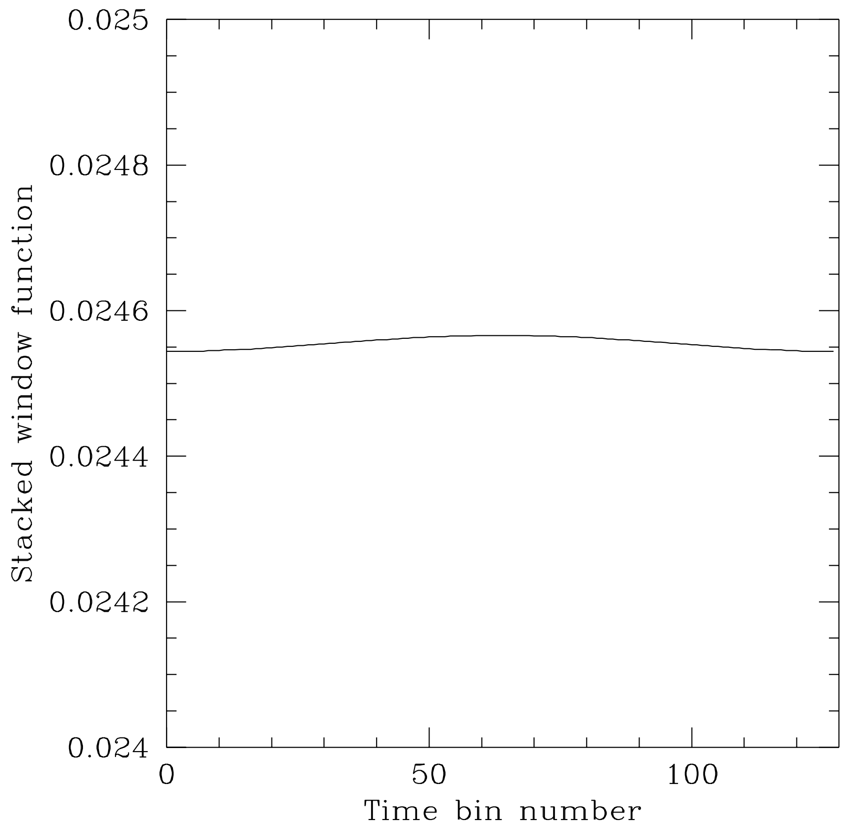

Figure 3: Window function used in the MWA fine PFB after stacking. Note the range of values on the vertical axis.

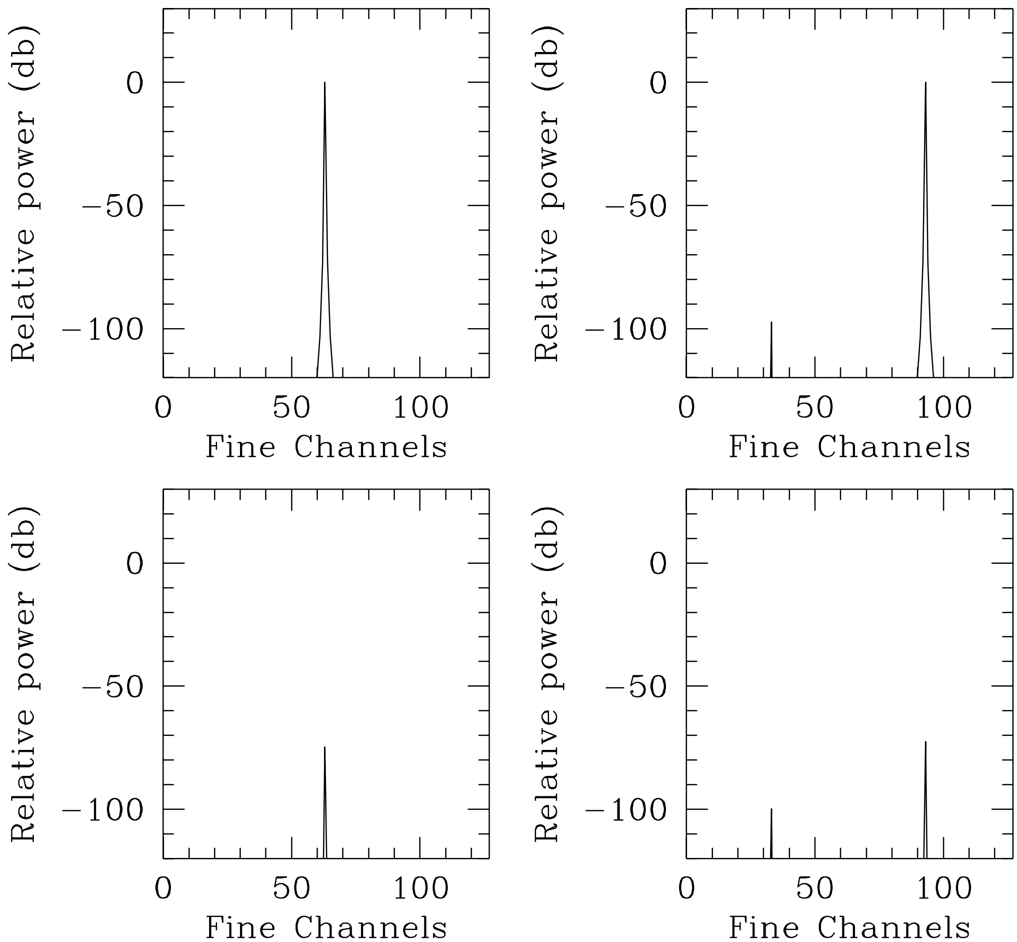

Figure 4: Results from a set of simulations of the MWA-LFD 32T PFBs. Each simulation assumes pure sinusoidal input; the frequencies are different in the different simuation cases. The plots show the power levels in the fine channels for a given coarse PFB channel (no. 197) regardless of the input frequency. The convention used here has the center frequency of a coarse channel at fine channel 63. The input frequencies are, relative to the center frequency of channel 197, are 0 MHz (top left), 2.56 MHz (bottom left), 0.30 MHz (top right), 2.86 MHZ (bottom right). Note that there are 128 fine channels that equally divide the 1.28 MHz coarse channel.

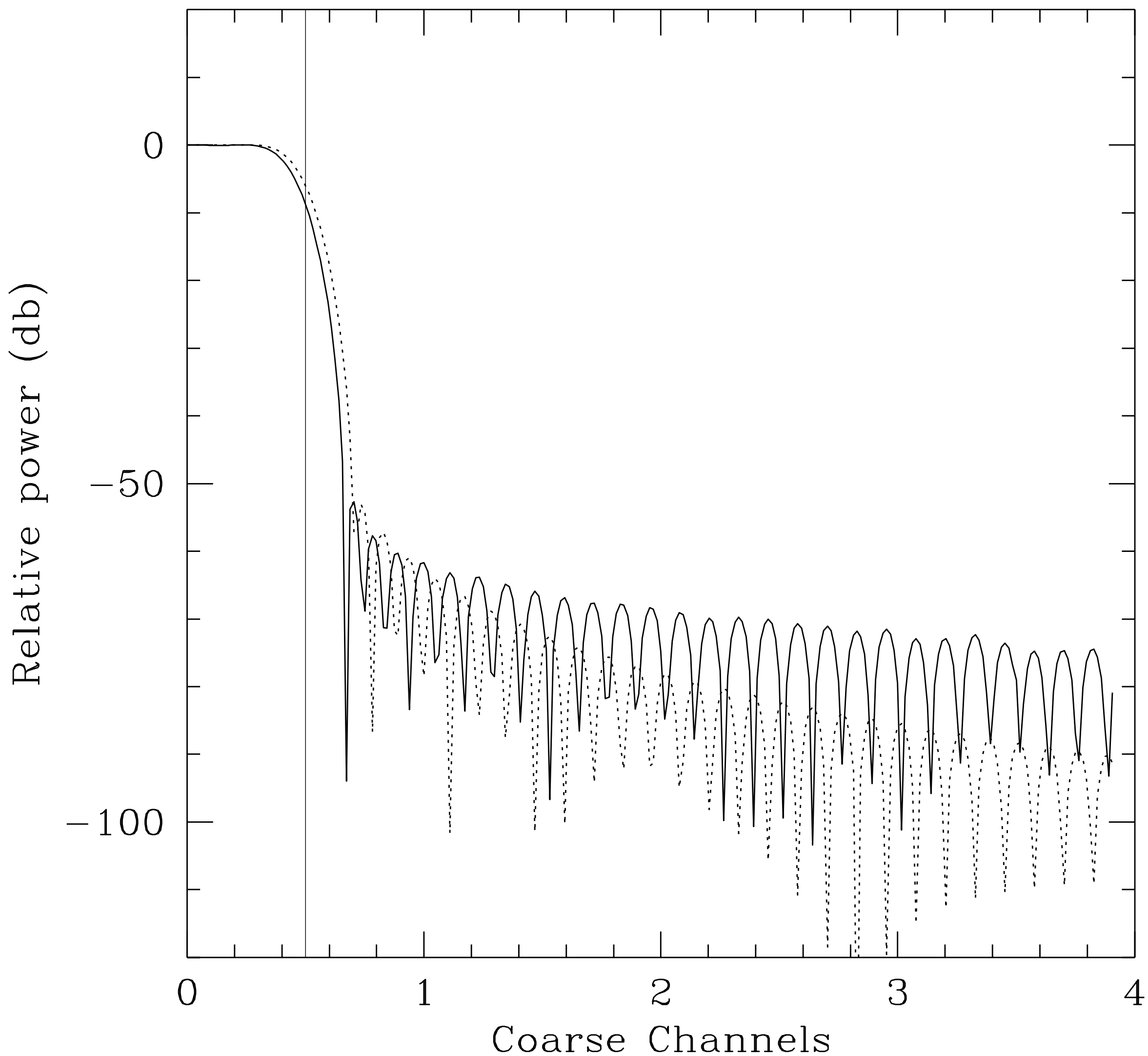

Figure 5: Power as a function of input frequency (shown here in terms of coarse channel number) as computed in the MWA-LFD 32T simulations. One coarse channel has a width of 1.28 MHz. There are 128 fine channels per coarse channel. The solid and dashed curves show the system response for two particular window functions. See the text for details.