Two-Element Interferometer Cable Ripple Analysis (Rev. 1)

Revision notes:

1. 7 Jul. 2016 - Started, A. Sutinjo
2. 18 Jul. 2016 - Rev. 1 started, A. Sutinjo. Corrected typos.

References:

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Context

We review a two-element interferometer without resorting to quasi-monochromatic assumption so as not to restrict ourselves to coherence time >> signal delay [1, 4]. The motivation is to understand the behavior of cable ripple seen by the interferometer [2, 3] and how to best measure and quantify that for engineering. We also seek to understand the interplay between cable ripple and low-order polynomial fitting employed in calibration.

Review of Theory

Response of a two-element interferometer

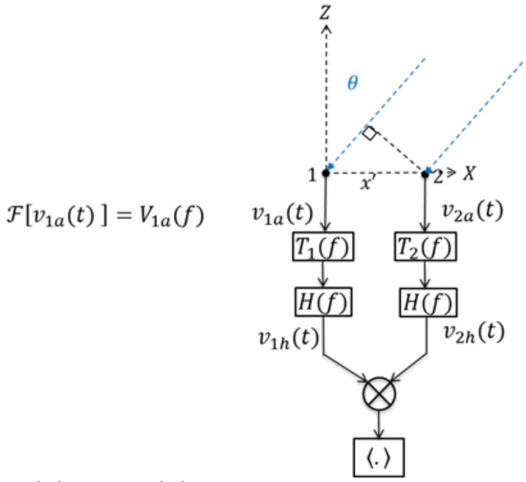


Fig. 1. A two element interferometer. $T_1(f)$ and $T_2(f)$ are the frequency responses of the long transmission lines. The frequency response of the electronic chains may be lumped into these parameters. $H(f)$ is the frequency response of the bandpass filter (identical in 1 and 2). $\mathcal{F}[]$ signifies fourier transformation.

The basis for treatment is found in random process/statistical communication theory textbooks [4, 5]. The treatment in radio astronomy context is given in [1]. It can be shown that the key equations are [6]:

- The output of the cross-correlator (Fig. 1):
$$G_{12}(\tau) = \langle v^*_{1h}(t) v_{2h}(t+\tau)\rangle = \int_{-\infty}^{\infty} T^*_1(f) T_2(f) |H(f)|^2 \langle V^*_{1a}(f) V_{2a}(f)\rangle e^{j2\pi f\tau} df \quad \dashrightarrow \text{Eq. (1)}$$
where $\langle V^*_{1a}(f) V_{2a}(f)\rangle$ is the cross-spectral density of the voltages present at the output of the antennas (or more precisely, present at the inputs of T_1 and T_2). Note that the time-domain voltages are represented as the positive complex pre-envelope [5] or complex wavefunction [4] such that the $\mathcal{F}[v(t)]$ is non-zero for positive frequencies only.
- The auto-correlation
$$G_{11}(\tau) = \langle v^*_{1h}(t) v_{1h}(t+\tau)\rangle = \int_{-\infty}^{\infty} |T_1(f)|^2 |H(f)|^2 \langle |V_{1a}|^2(f)\rangle e^{j2\pi f\tau} df \quad \dashrightarrow \text{Eq. (2)}$$
similarly for chain #2.

Special cases:

Single-source radiation

Consider a *special* case of a *single* source at direction θ. For a 1-D sky, we obtain

$$\langle V_{1a}^*(f) V_{2a}(f) \rangle = e^{j2\pi f \tau_s} \langle |V(\hat{k}_x, f)|^2 \rangle \quad \dashrightarrow \text{Eq. (3)}$$

where $\tau_s = \frac{x' \sin\theta}{c} = \frac{x' \hat{k}_x}{c}$ is the space delay and $\langle |V_0(\hat{k}_x, f)|^2 \rangle$ is the power spectral density of the source at $\hat{k}_x = \sin\theta$.

Uncorrelated sources from multiple directions

If we assume the interferometer is illuminated by uncorrelated sources from multiple (discrete) directions, the right-hand-side of Eq. (3) becomes

$$\langle V_{1a}^*(f) V_{2a}(f) \rangle = \sum_{i=1}^{N} e^{j2\pi f \frac{x'}{c} \hat{k}_{xi}} \langle |V(\hat{k}_{xi}, f)|^2 \rangle \quad \dashrightarrow \text{Eq. (3b)}$$

In the limit of continuous distribution of sources ($\hat{k}_x$) and continuous sampling on the ground ($f \frac{x'}{c} = x'/\lambda$), Eq.(3b) becomes a fourier transform [1].

Connection to Electrical Engineering

Eq. (1, 2) above are quite instructive. They tell us that the output of the correlator is directly related to the *frequency* responses of the cascaded stages. This is good, because

- Frequency-domain analyses and formulas for tranmission lines (TL) [7, 8] are useable here.
- The transfer function of a long cable is measurable with a frequency swept device such as a vector network analyzer (VNA), bearing in mind that the output frequency lags the input frequency by

  $$\Delta f = \frac{df}{dt} \tau_{\text{cable}} \quad \dashrightarrow \text{Eq. (4)}$$

  where $\frac{df}{dt}$ is the sweep rate and τ_{cable} is the cable delay; the sweep rate must be such that $\Delta f <<$resolution (or IF) bandwidth of the VNA. For a 150m RG6 cable with velocity factor of 0.66c and 0.76μs delay, the sweep rate should be << 13GHz/s for a 10kHz IF bandwidth (BW)

Cable Response

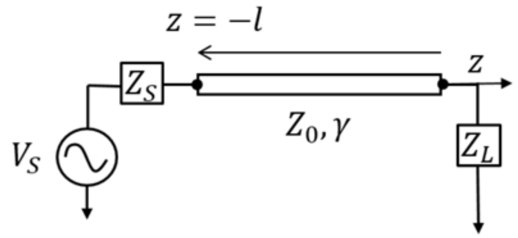


Fig. 2. A lossy transmission line (TL).

The point of this exercise is to deal with concerns regarding long cable ripples. These ripples are due to partial standing waves in the transmission line (Fig. 2) described in [7, 8]

$$V(z) = \frac{V_S Z_0}{Z_0 + Z_S} e^{-\gamma l} \frac{e^{-\gamma z} + \Gamma_L e^{\gamma z}}{1 - \Gamma_L \Gamma_S e^{-2\gamma l}} \dashrightarrow \text{Eq. (5)}$$

where $\gamma = \alpha + j\beta$ is the complex propagation constant, $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$ and $\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0}$ are the load and source reflection coefficients, respectively.

We are interested in the transfer function $T(f)$ of the TL which relates the output to input voltages. The voltage at the load is

$$V(z = 0) = \frac{V_S Z_0}{Z_0 + Z_S} e^{-\gamma l} \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_S e^{-2\gamma l}} \dashrightarrow \text{Eq. (6)}$$

The voltage at the input is

$$V(z = -l) = \frac{V_S Z_0}{Z_0 + Z_S} e^{-\gamma l} \frac{e^{\gamma l} + \Gamma_L e^{-\gamma l}}{1 - \Gamma_L \Gamma_S e^{-2\gamma l}} \dashrightarrow \text{Eq. (7)}$$

Now, the question is: which do we call $T(f)$? $V(0)/V(-l) = \frac{1+\Gamma_L}{e^{\gamma l} + \Gamma_L e^{-\gamma l}}$ or $V(0)/V_S$? Furthermore, what is the physical meaning of Z_S? Consider an idealized diagram representing the antenna, low-noise amplifier (LNA) and the TL in Fig. 3. We assume that the LNA is an ideal buffer amplifier. The desired ratio is $V(z=0)/V_{OC}$; hence,

$$T(f) = V(0)/V_S = \frac{Z_0}{Z_0 + Z_S} e^{-\gamma l} \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_S e^{-2\gamma l}} \dashrightarrow \text{Eq. (8)}$$

where $Z_S = Z_{o:LNA}$ is the LNA output impedance. The *strongest* frequency dependence in Eq. (8) is $\beta(f) = \frac{2\pi f}{v_p}$ in $\gamma(f) = \alpha + j\beta(f)$, where v_p is the phase velocity in the TL.

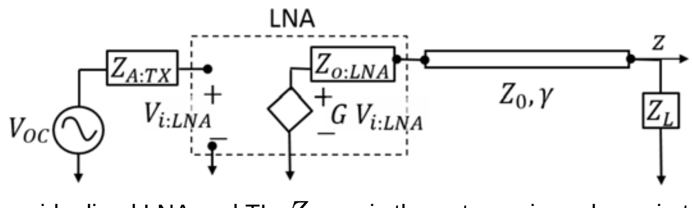


Fig. 3. Antenna connected to an idealized LNA and TL. $Z_{A:TX}$ is the antenna impedance in transmit mode and V_{OC} is the antenna open circuit voltage in receive mode.

The model in Fig. 3 may be refined to include the ratio between V_{OC} and $V_{i:LNA}$ due to realistic LNA input impedance which will introduce an additional transfer function. Alternatively, this affect may be accounted by calculating antenna response including the LNA input impedance. We will not pursue that level of detail here, however.

This concludes our review. We will now move to computer simulation.

In [1]:
```
# Define functions to calculate cable transfer function
#IMPORTANT: my Kernel is Python 3.5
%matplotlib notebook
#%matplotlib inline
import matplotlib.pyplot as plt
import numpy as np

#housekeeping
eps=np.finfo(float).eps #floating resolution

TL_c=299792458 # velocity of light in vacuum

#dB conversion-------------------------------

def V_to_dB(Vratio): #convert voltage ratio to dB
    return 20*np.log10(abs(Vratio+eps))

def P_to_dB(Pratio): #convert power ratio to dB
    return 10*np.log10(abs(Pratio+eps))
#---------------------------------------------

def TL_gamma(Zl,Zo):
    """
    voltage reflection coefficient.
    Zl: load impedance (cplx, Ohms)
    Zo: TL characteristic impedance (generally real, Ohms)
    """
    return (Zl-Zo)/(Zl+Zo)

print(V_to_dB(TL_gamma(73,75)))
```

```
-37.3846343946
```

In [2]:
```
def TL_beta(vp,f):
    """
    function to calculate phase constant \beta at frequency f (real)
    at a given phase velocity vp (real)
    """
    return 2*np.pi*f/vp

def TL_Npm(dB_loss,TL_length_m): #convert dB loss in x m to alpha in Np/m
    return dB_loss/(20*TL_length_m*np.log10(np.exp(1)))

print(TL_beta(TL_c*0.6,np.linspace(100e6,120e6,10)))
print(TL_Npm(1,1))
```

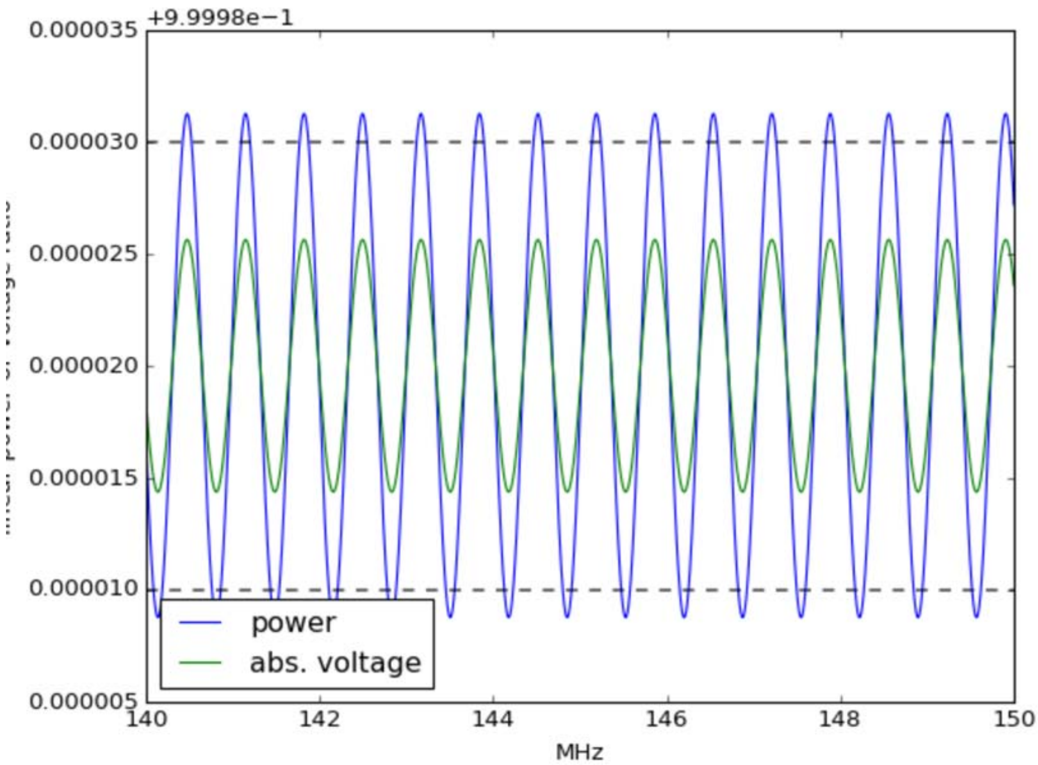
```
[ 3.49307504  3.57069893  3.64832282  3.72594671  3.8035706   3.88119449
  3.95881837  4.03644226  4.11406615  4.19169004]
0.11512925465
```

In [3]:
```
def TL_transfer(Zo,Zs,Zl,alp,bet,l):
    """
    Tl transfer function per Eq. (8)
    Zs: source impedance (cplx, Ohms)
    alp: attenuation constant
    bet: phase constant
    l: physical length of the TL in m
    """
    GamL=TL_gamma(Zl,Zo)
    GamS=TL_gamma(Zs,Zo)
    g=alp+bet*1j #propagation constant
    return (Zo/(Zo+Zs))*np.exp(-g*l)*(1+GamL)/(1-GamS*GamL*np.exp(-2*g*l))
```

Example 1: Cable Ripple

In [4]:
```
#Example 1: RG6 cable ripple
Zo=75
Zl=73 #hypothetically well matched load
Zs=77 # and source
TL_len=150
eps_r=2.2 #relative permittivity of Polyethylene
freq=np.linspace(140,150,2000)*1e6 #frequency
bet=TL_beta(TL_c/(eps_r)**0.5,freq)
alp=TL_Npm(10,100) #10 dB loss per 100 m, approx. for RG 6 at couple hundred MHz

print("Gamma_L (dB)=","%.2f" %float(V_to_dB(TL_gamma(Zl,Zo))))
print("Gamma_S (dB)=","%.2f" %float(V_to_dB(TL_gamma(Zs,Zo))))

T_test=TL_transfer(Zo,Zs,Zl,alp,bet,TL_len)

delta_BM=1e-5; #Barry-Morales limit as per [2]
plt.figure(1)
line_power, = plt.plot(freq/1e6,abs(T_test)**2/np.mean(abs(T_test))**2, label='p
ower')
line_voltage, = plt.plot(freq/1e6,abs(T_test)/np.mean(abs(T_test)), label='abs.
voltage')
plt.plot(freq/1e6,1+delta_BM*np.ones(np.size(freq)),'k--')
plt.plot(freq/1e6,1-delta_BM*np.ones(np.size(freq)),'k--')
plt.legend(handles=[line_power, line_voltage],loc=3)
plt.xlabel('MHz')
plt.ylabel('linear power or voltage ratio')
```

```
Gamma_L (dB)= -37.38
Gamma_S (dB)= -37.62
```



Out[4]: <matplotlib.text.Text at 0x74bfb70>

Analyzing the ripple for frequency-independent $Z_{S,L,0}$ and α

In Example 1 above, we made $Z_{S,L,0}$ and α frequency independent and only β frequency dependent which applies in practice assuming $\beta(f)$ is the *dominant* cause for ripple. For this *special case*, the shape of the cable response is fully described by:

$$T'_{\beta(f)} = \frac{1}{1-\Gamma_L\Gamma_S e^{-2\gamma l}} = \sum_{n=0}^{\infty}(\Gamma_L\Gamma_S e^{-2\gamma l})^n \quad \dashrightarrow \text{Eq. (9)}$$

The geometric series expansion [7] is possible since for physical reflections $\left|\Gamma_L\Gamma_S e^{-2\gamma l}\right| < 1$. Note that arbitrary cable ripples may be modeled with a single sinusoidal function in the *denominator* of Eq. (9).

In the special case of small reflections, i.e., $\left|\Gamma_L\Gamma_S e^{-2\gamma l}\right| << 1$,

$$T'_{\beta(f)} \approx 1 + \Gamma_L\Gamma_S e^{-2\alpha l}e^{-2j\beta l} \quad \dashrightarrow \text{Eq. (10)}$$

such that

$$\left|T'_{\beta(f)}\right| \approx \sqrt{1+2|\Delta|\cos(2\beta l-\angle\Delta)+|\Delta|^2} \approx 1+|\Delta|\cos(2\beta l-\angle\Delta)+\frac{|\Delta|^2}{2} \quad \dashrightarrow \text{Eq. (11)}$$

where $|\Delta| = |\Gamma_L\Gamma_S|\, e^{-2\alpha l} << 1$ and $\angle\Delta = \angle(\Gamma_L\Gamma_S)$. The equation above suggests that for small reflections:

- We can fit the cable ripple with a single sinusoid $\cos(2\beta l)$ for *both* voltage and power quantities
- The cable ripple "period" in frequency is given by $f = \frac{v_p}{2l} = \frac{1}{2\tau_c}$, where τ_c is the one-way cable delay. Preventing ripples with periods shorter than 8 MHz [2] translates to keeping $\tau_c < 62.5$ ns which for an RG6 cable is less than 12.4 m!
- The ripple in $\left|T'_{\beta(f)}\right|^2$ is twice as much as in $\left|T'_{\beta(f)}\right|$ as expected from $(1 \pm \Delta v)^2 = 1 \pm 2\Delta v + \Delta v^2$.
- Taking the 10^{-5} in [2] as the ripple limit (Δ) for $\left|T'_{\beta(f)}\right|$ suggests $\Delta = |\Gamma_L\Gamma_S|\, e^{-2\alpha l} < 10^{-5}$, or in dB:

$$|\Gamma_L|\ (dB) + |\Gamma_S|\ (dB) - 2\times\text{cable\_loss}\ (dB) < -100\ (dB) \quad \dashrightarrow \text{Eq. (12)}$$

For a 150 m long RG6 cable at $\sim$ 150 MHz, the cable loss is 15 dB. The requirement above may be met with 35 dB return losses at the source and load, which is more stringent than the typical 20-30 dB return loss offered by COTS amplifier amplifier (http://pdf1.alldatasheet.com/datasheet-pdf/view/174714/SIRENZA/CGB-1089Z.html) or high-precision cable (https://www.belden.com/docs/upload/NP233.pdf). RF isolators (https://raditek.com/IC_COAXIAL/25-299/100-199MHz/RADCorI-117-175M-S_N_-200WR-Generic-b.pdf) offer $\sim$20 dB return loss which is not helpful.

How about fiber optics?

Assuming an intensity modulated RF-over-Fiber (RFoF) link [10], the optical power ripple is commensurate to the RF amplitude voltage ripple seen by the link. As suggested by Eq. (11), the ripple is $\sim 2\Delta$ such that

$$|\Gamma_L|\ (dB) + |\Gamma_S|\ (dB) - 2\times\text{fiber\_optic\_loss}\ (dB) < -106\ (dB) \quad \dashrightarrow \text{Eq. (13)}$$

For a single mode fiber at 1310 nm, the optical loss is typically ~0.5 dB/km (http://www.thefoa.org/tech/loss-est.htm). Connecting an optical isolator (https://www.thorlabs.com/thorcat/17200/IO-H-1310APC-AutoCADPDF.pdf) rated for 29 dB minimum isolation and >50 dB return loss to a well-matched (return loss > 24 dB) RFoF transmitter and receiver link may do the job (2x(29+24) = 106 dB).

Summary of uncalibrated cable response

Achieving 10^{-5} cable ripple with an uncalibrated system:

- cannot be consistently expected with RF cables. It may work with very lossy cables, but this is not desirable.
- appears feasible in fiber optics by employing optical isolators and well-matched RFoF links, but further verification is needed such as including the effect of splices and/or connectors (http://www.thefoa.org/tech/ref/testing/test/reflectance.html) etc.

Next, we consideration calibration with low-order polynomial fitting.

Fitting Cable Ripple with Low-Order Polynomial

We now look at applying the Trott-Wayth (TW) low-order polynomial fitting as discussed in [3]. We assume a time-invariant system for now. This allows us to perform deterministic polynomial expansion. Obviously, when working with a real signal we will need to use a least-squares approach.

For small ripples, we use Eq. (11)

$$T'_{\beta(f)} \approx 1 + |\Delta|\, e^{j\angle\Delta} e^{-j2\beta l}$$

Let $e^{-j2\beta l} = e^{-j2\pi(f_0+\delta_f)2\tau_c}$ where f_0 is a nominal center frequency. Using Maclaurin series expansion

$$e^{-jz} = \sum_{n=0}^{\infty} \frac{(-jz)^n}{n!} \approx 1 - \frac{z^2}{2} + \frac{z^4}{24} + j\left(-z + \frac{z^3}{6} - \frac{z^5}{120}\right) \quad \dashrightarrow \text{Eq. (14)}$$

Using a 3rd-order polynomial fitting, we are left with a residue

$$Res(e^{-jz}) \approx \frac{z^4}{24} - j\frac{z^5}{120} \quad \dashrightarrow \text{Eq. (15)}$$

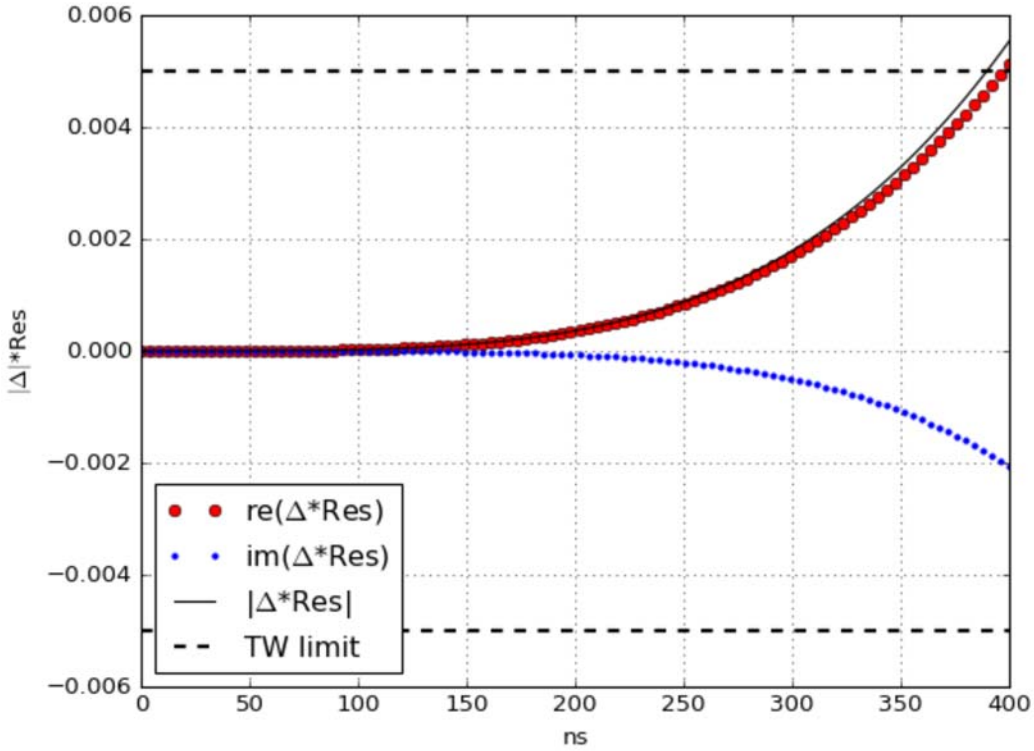
Following [3], we calculate the residue of a 3rd order polynomial approximation at the edges of a coarse channel ($B_c = 769$ kHz/ch.). Let $z = 2\pi\delta_f 2\tau_c$ and f_0 be the midpoint of the central coarse channel. Eq. (15) suggests that the maximum residue occurs at the band edges at $\delta_f = \pm 0.5 B_c = \pm 0.385$ MHz.

TW Limit: Example 2

Example 2 below calculates the exact band-edge residue as a function of cable delay for 3rd order polynomial approximation of cable ripple for a lossless cable with 20 dB source and load return loss ($|\Delta| = 10^{-2}$, consistent with "good" RF match). The residue is compared to the 5×10^{-3} TW limit suggested in [3].

In [5]:
```
def TL_fit3(delf_max,tau_c):
    #approximate residue in ripple fitting using 3rd-order polynomial as a funct
ion of one-way cable delay
    #del_f: maximum delta freq from center (MHz)
    #tau_c: one-way cable delay (microseconds)
    z=2*np.pi*delf_max*2*tau_c
    fit3=1-z**2/2+1j*(-z+z**3/6) #3rd order fit
    res=np.exp(-1j*z)-fit3 #exact residue
    return (fit3,res)

#Example 2: residue at coarse channel band edge as a function of cable delay
delf_max=0.769*0.5 #in MHz
tau_c=np.linspace(0,0.4,100) #in microseconds
Gl_Gs=0.1*0.1#|Gamma_l||Gamma_s|: 20 dB load and source return loss, 0 dB cable
loss
fit_res=TL_fit3(delf_max,tau_c) #[0] is fit, [1] is res
del_res=fit_res[1]*Gl_Gs #delta*residue

print('Delta=',"%e" % float(Gl_Gs))

plt.figure(2)
#plot residue
line_real_res,=plt.plot(tau_c*1e3,np.real(del_res),'ro',label=r're($\Delta$*Res)
')
line_imag_res,=plt.plot(tau_c*1e3,np.imag(del_res),'b.',label=r'im($\Delta$*Res)
')
line_abs_res,=plt.plot(tau_c*1e3,abs(del_res),'k-',label=r'|$\Delta$*Res|')

delta_TW=0.005 #most stringent Trott-Wayth limit
line_lim_res,=plt.plot(tau_c*1e3,delta_TW*np.ones(np.size(tau_c)),'k--',linewidt
h=2,label='TW limit')
plt.plot(tau_c*1e3,-delta_TW*np.ones(np.size(tau_c)),'k--',linewidth=2)

plt.xlabel('ns')
plt.ylabel(r'$|\Delta|$*Res')
plt.grid()
plt.legend(handles=[line_real_res, line_imag_res,line_abs_res,line_lim_res],loc=
3)
```

```
Delta= 1.000000e-02
```




```
Out[5]: <matplotlib.legend.Legend at 0x76c0fd0>
```

Remarks and design implications of TW limit:

- Reasonably well-matched lossless cables (20 dB source and load return loss) up to ~ 390 ns may be calibrated (modelled) with a 3rd order polynomial fit
- Assuming optical index of 1.467 (single mode fiber, SMF (http://ece466.groups.et.byu.net/notes/smf28.pdf)), this delay translates to ~ 80 m.
- Longer cables (or uncalibrated cables) must meet Eq. (12, 13), adapted to the TW limit of 5×10^{-3}:

$$|\Gamma_L| \ (dB) + |\Gamma_S| \ (dB) - 2 \times \text{cable\_loss} \ (dB) < -66 \ (dB) \quad \dashrightarrow \text{Eq.} \ (16)$$
$$|\Gamma_L| \ (dB) + |\Gamma_S| \ (dB) - 2 \times \text{fiber\_optic\_loss} \ (dB) < -72 \ (dB) \quad \dashrightarrow \text{Eq.} \ (17)$$

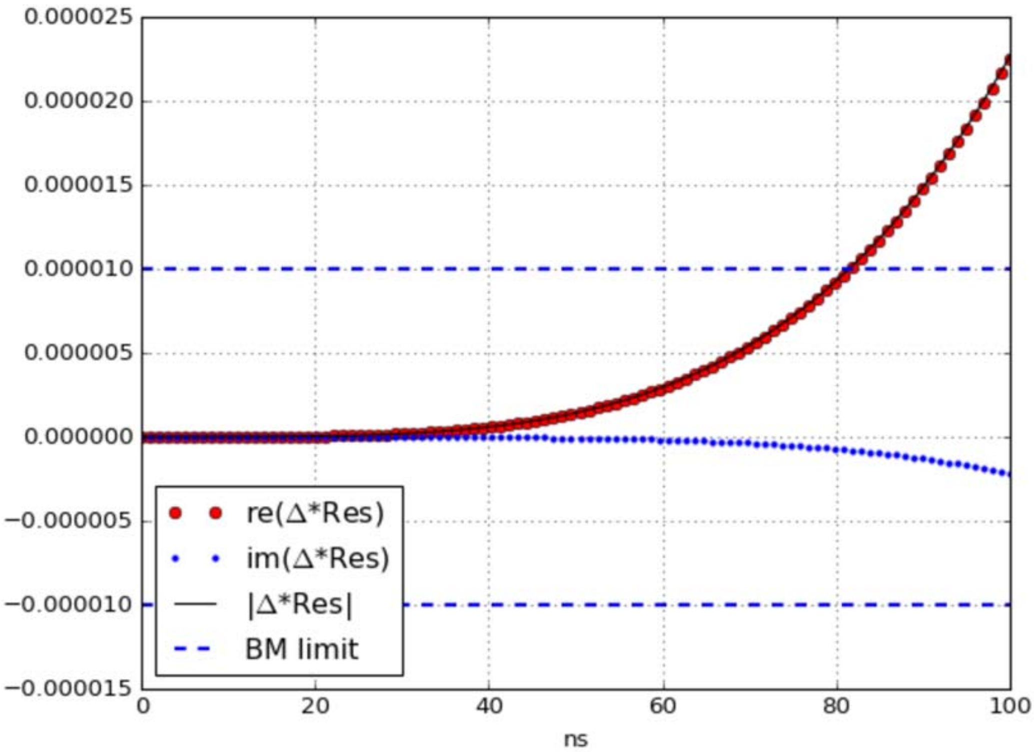
Taking Eq. (17) and assuming a lossless cable suggests 36 dB source and and load reflection coefficient (max.) which seems to be near the limit of what is achieveable without optical isolator (TBC).

BM Limit: Example 3

Example 3 below calculates the exact band-edge residue as a function of cable delay for 3rd order polynomial approximation of cable ripple for a lossless cable with 20 dB source and load return loss. The residue is compared to the 10^{-5} BM limit suggested in [3].

In [6]:
```
#Example 3: residue at coarse channel band edge as a function of cable delay
#initialize
tau_c=[]
fit_res=[]
del_res=[]
#recalculate
tau_c=np.linspace(0,0.1,100) #in microseconds
fit_res=TL_fit3(delf_max,tau_c) #[0] is fit, [1] is res
del_res=fit_res[1]*Gl_Gs #delta*residue

print('Delta=',"%e" % float(Gl_Gs))

plt.figure(3)
#plot residue
line_real_res,=plt.plot(tau_c*1e3,np.real(del_res),'ro',label=r're($\Delta$*Res)
')
line_imag_res,=plt.plot(tau_c*1e3,np.imag(del_res),'b.',label=r'im($\Delta$*Res)
')
line_abs_res,=plt.plot(tau_c*1e3,abs(del_res),'k-',label=r'|$\Delta$*Res|')

line_lim_res,=plt.plot(tau_c*1e3,delta_BM*np.ones(np.size(tau_c)),'b--',linewidt
h=2,label='BM limit')
plt.plot(tau_c*1e3,-delta_BM*np.ones(np.size(tau_c)),'b--',linewidth=2)

plt.xlabel('ns')
plt.ylabel(r'$|\Delta|$*Res')
plt.grid()
plt.legend(handles=[line_real_res, line_imag_res,line_abs_res,line_lim_res],loc=
3)
```

```
Delta= 1.000000e-02
```



Out[6]:
```
<matplotlib.legend.Legend at 0x7726f98>
```

Remarks on the BM limit:

- Reasonably well-matched lossless cables (20 dB source and load return loss) up to ~ 81 ns may be calibrated (modelled) with a 3rd order polynomial fit. This is consistent with the 62.5 ns cable delay resulting in 8 MHz ripple period [2]; in fact the 0.5*BM limit line intersects the real part of the residue at 68 ns in Fig. 3.

TW vs. BM Limit Summary

- Eq. (12,13) and (16,17) are useful in link design as per BM and TW limits, respectively.
- Assuming a lossless cable, the TW limit calls for very-well matched source and load while BM requires extremely well-matched source and load.
- The TW limit may be achievable without optical isolators with great care in the design. Meeting the BM limit generally requires optical isolators.

Correlator output of an interferometer with a long cable

Let $H(f)$ be an ideal ("brickwall") bandpass filter centered at f_0 with bandwidth B and $T_1(f) = 1$. For a single source illumination (from $\hat{k}_x$), the output the correlator is given by

$$G_{12}(\tau) = \int_{f_0-B/2}^{f_0+B/2} T_2(f)\langle|V(\hat{k}_x, f)|^2\rangle e^{j2\pi f\tau_s} e^{j2\pi f\tau} df \dashrightarrow \mathrm{Eq.}\,(18)$$

Again, assuming the frequency dependence in $T_2(f)$ is dominated by $\beta(f)$,

$$T_2(f) = C\frac{e^{-j2\pi f\tau_c}}{1 - \Delta e^{-j2\pi f 2\tau_c}}$$

where $C = \frac{Z_0}{Z_0+Z_S}e^{-\alpha l}(1+\Gamma_L)$ and $\Delta = \Gamma_L\Gamma_S e^{-2\alpha l}$ are assumed independent of frequency. Further, again assuming small reflections, we obtain

$$T_2(f) \approx C(1 + \Delta e^{-j2\pi f 2\tau_c})e^{-j2\pi f\tau_c}$$

Hence, the output of a two-element correlator with a long cable and small reflections illuminated by a single source is

$$G_{12}(\tau) \approx C\int_{f_0-B/2}^{f_0+B/2} \langle|V(\hat{k}_x, f)|^2\rangle(1 + \Delta e^{-j2\pi f 2\tau_c})e^{j2\pi f(\tau_s-\tau_c+\tau)} df \dashrightarrow \mathrm{Eq.}\,(19)$$

To clarify the effect of cable ripple alone, consider a special case with flat source spectrum, $\langle|V(\hat{k}_x, f)|^2\rangle$=constant, and $\tau = 0$. Let $\delta_\tau = \tau_s - \tau_c$, we get

$$G_{12}(0) \sim \int_{f_0-B/2}^{f_0+B/2}(1 + \Delta e^{-j2\pi f 2\tau_c})e^{j2\pi f\delta_\tau} df = Be^{j2\pi f_0\delta_\tau}\left(\frac{\sin(\pi B\delta_\tau)}{\pi B\delta_\tau} + e^{-j2\pi f_0 2\tau_c}\Delta\frac{\sin(\pi B(2\tau_c - \delta_\tau))}{\pi B(2\tau_c - \delta_\tau)}\right) \dashrightarrow$$

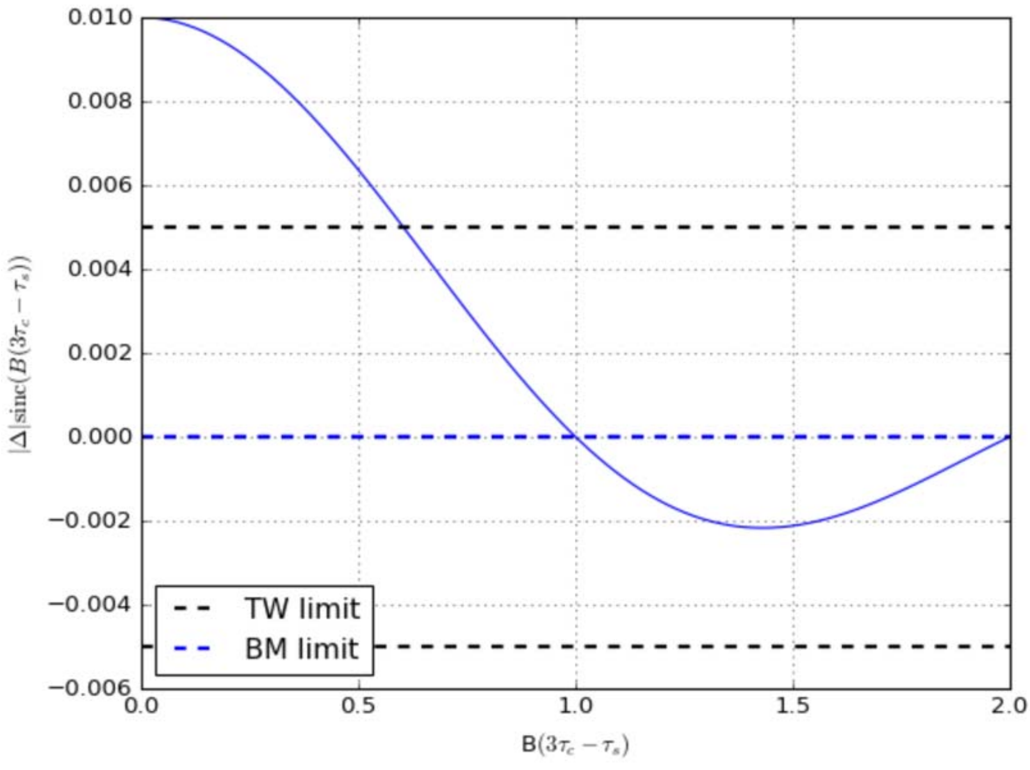
Hence, for $B << 1/(2\tau_c - \delta_\tau)$, the ripple $|\Delta|$ is fully seen at the output of the correlator as a function of center frequency f_0. However, as $B(2\tau_c - \delta_\tau)$ becomes larger, *less* ripple is produced at the output of the correlator as per $\mathrm{sinc}(B(2\tau_c - \delta_\tau) = \mathrm{sinc}(B(3\tau_c - \tau_s))$.

This is relevant for engineering as demonstrated in Example 4 below. Again, taking $\Delta = 10^{-2}$ commensurate to 20 dB source and load return loss and a lossless cable:

- The ripple is less than TW limit for $B(3\tau_c - \tau_s) > 0.6$. For $x' = 3$ km such that the maximum $\tau_s = 10\ \mu s$ and B=4.56 kHz [3] ($0.6/B = 131\ \mu s$), this translates to cable delay of $\tau_c > 47\mu s$ or $l \sim >$9.5 km for a SMF. Since this distance is significantly farther than 3 km, it is of limited practical benefit.
- The ripple is less than BM limit for $B(3\tau_c - \tau_s) > \frac{10^{-2}}{2\pi 10^{-5}} = 159$, which definitely will not provide practical benefit.

In [7]:
```
#Example 4: correlator output at tau=0 for a single source with flat spectrum an
d T_{1}=1
Bt=np.linspace(0,2,100) #BW*tau_c

print('Delta=',"%e" % float(Gl_Gs))

plt.figure(4)
line_sinc,=plt.plot(Bt,Gl_Gs*np.sinc(Bt))
#TW limit
line_lim_res_TW,=plt.plot(Bt,delta_TW*np.ones(np.size(Bt)),'k--',linewidth=2,lab
el='TW limit')
plt.plot(Bt,-delta_TW*np.ones(np.size(Bt)),'k--',linewidth=2)
#BM limit
line_lim_res_BM,=plt.plot(Bt,delta_BM*np.ones(np.size(Bt)),'b--',linewidth=2,lab
el='BM limit')
plt.plot(Bt,-delta_BM*np.ones(np.size(Bt)),'b--',linewidth=2)
plt.grid()
plt.xlabel(r'B$(3\tau_c-\tau_s)$')
plt.ylabel(r'$|\Delta|\mathrm{sinc}(B(3\tau_c-\tau_s))$')
plt.legend(handles=[line_lim_res_TW,line_lim_res_BM],loc=3)
```

```
Delta= 1.000000e-02
```



Out[7]: <matplotlib.legend.Legend at 0x7795ef0>

Conclusion

- BM ripple limit of 10^{-5} suggests that cables with delays of $\sim$62.5 ns or greater must not exceed the limit.
- TW ripple limit of 5×10^{-3} suggests that cables with delays of $\sim$390 ns or greater must not exceed the limit.
- For baselines of up to a few kms and fine channel of a $\sim$few kHz, we cannot rely on the fine channel filtering and correlation to "smooth-out" the ripple down to the limit.
- The TW limit may be achievable without optical isolators with great care in the design. Meeting the BM limit generally requires optical isolators.

Next steps:

- Collect measured data with RFoF link. Apply low-order polynomial fitting and compute the residue. Practice this in the lab, then repeat at the MRO (Budi, Adrian, student).
- Determine test equipment (VNA/ENA) settings appropriate to measure the low-level ripples, taking into account consideration from correlation's perspective (Budi, Adrian, student).
- Verify the ripple formulas and calculations for RFoF link (Budi, Adrian, student)
- Suggestions?

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